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Quasilinear QMV Algebras

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We investigate quasilinear and weakly linear QMV algebras as a generalization of the algebraic structure of all effects of a Hilbert space and we study the varieties generated by these classes. Finally, we prove some results concerning locally finite and Archimedean QMV algebras.

INTRODUCTION

QMV (quantum MV) algebras were introduced in Giuntini (n.d.) as a generalization both of MV (multivalued) algebras and of the structure determined by the class of all *effects* of a Hilbert space. In this paper we will investigate some structural properties of the QMV algebras based on effects (standard QMV algebras) and we will try to generalize some results based on these "concrete" structures to abstract QMV algebras. We will show that some convenient weakenings of important properties of the standard MV algebra based on the real interval [0, 1] hold also in the standard QMV algebras. In particular, standard QMV algebras turn out to be quasilinear, locally finite, and Archimedean.

1. QUANTUM MV ALGEBRAS

Definition 1.1. A *quantum MV algebra* (*QMV algebra*) is a structure $\mathcal{M} = \langle M, \oplus, {}^*, \mathbf{1}, \mathbf{0}\rangle$, where M is a nonempty set, $\mathbf{0}$ and $\mathbf{1}$ are constant elements of M, $\oplus$ is a binary operation, and * is a unary operation, satisfying the following axioms [where $a \odot b := (a^* \oplus b^*)^*$, $a \Cap b := (a \oplus b^*) \odot$

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b, and $a \Cup b := (a \odot b^*) \oplus b$]:

(QMV1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(QMV2) $a \oplus \mathbf{0} = a$.
(QMV3) $a \oplus b = b \oplus a$.
(QMV4) $a \oplus \mathbf{1} = \mathbf{1}$.
(QMV5) $(a^*)^* = a$.
(QMV6) $\mathbf{0}^* = \mathbf{1}$.
(QMV7) $a \oplus a^* = \mathbf{1}$.
(QMV8) $a \Cup (b \Cap a) = a$.
(QMV9) $(a \Cap b) \Cap c = (a \Cap b) \Cap (b \Cap c)$.
(QMV10) $a \oplus (b \Cap (a \oplus c)^*) = (a \oplus b) \Cap (a \oplus (a \oplus c)^*)$.
(QMV11) $a \oplus (a^* \Cap b) = a \oplus b$.
(QMV12) $(a^* \oplus b) \Cup (b^* \oplus a) = \mathbf{1}$.

We assume $\odot$ to be more binding than $\oplus$. (QMV8) and (QMV9) represent a weak formulation of the absorption and of the associativity laws, respectively. Generally, $\Cap$ and $\Cup$ are not lattice-theoretic operations. (QMV10) and (QMV11) represent a kind of conditional distributivity law of $\oplus$ over $\Cap$.

Definition 1.2. An *MV algebra* is an algebraic structure $\mathcal{M} = \langle M, \oplus, {}^*, \mathbf{1}, \mathbf{0}\rangle$ that satisfies (QMV1)–(QMV7) and the following condition:

(LA) $(a \odot b^*) \oplus b = (b \odot a^*) \oplus a$ (*Łukasiewicz axiom*).

As proved in Chang (1957), any MV algebra is a QMV algebra.

Definition 1.3. Let $\mathcal{M}$ be a QMV algebra. For all $a, b \in M$

$$a \preceq b \quad \text{iff} \quad a = a \Cap b$$

Example 1.1 (*Standard MV algebra*). Let [0, 1] be the unit real interval. For all $a, b \in [0, 1]$, let

$$a \oplus b := \text{Min}(\{a + b, 1\}) \qquad (\textit{truncated sum})$$

and

$$a^* := 1 - a$$

The structure $\mathcal{M}_{[0,1]} = \langle [0, 1], \oplus, {}^*, 1, 0\rangle$ is an MV algebra, called *standard MV algebra*. It turns out that the relation $\preceq$ coincides with the restriction to [0, 1] of the usual order of $\mathbb{R}$. Consequently, $\mathcal{M}_{[0,1]}$ is *linear*, i.e., $\forall a, b \in [0, 1]$: $a \preceq b$ or $b \preceq a$.

It turns out that $a \odot b = \text{Max}(\{a + b - 1, 0\})$, $a \Cap b = \text{Min}(\{a, b\})$, and $a \Cup b = \text{Max}(\{a, b\})$.

Example 1.2 (*Standard QMV algebra*). Let $E(\mathcal{H})$ be the class of all effects of a Hilbert space $\mathcal{H}$. $E(\mathcal{H})$ coincides with the class of all bounded linear operators between $\mathbb{0}$ and $\mathbb{1}$, where $\mathbb{0}$ and $\mathbb{1}$ are the null and the identity operator, respectively. The operation $\oplus$ is defined as follows, for any E, $F \in E(\mathcal{H})$:

$$E \oplus F := \begin{cases} E + F & \text{if} \quad E + F \in E(\mathcal{H}) \\ \mathbb{1} & \text{otherwise} \end{cases}$$

where $+$ is the usual operator-sum.

$$E^* := \mathbb{1} - E$$

One can prove (Giuntini, n.d.) that the structure $\mathscr{E}(\mathcal{H}) = \langle E(\mathcal{H}), \oplus, {}^*, \mathbb{1}, \mathbb{0}\rangle$ is a QMV algebra, called *standard QMV algebra*. The structure $\mathscr{E}(\mathcal{H})$, however, is not an MV algebra, since it violates the crucial axiom (LA), which is responsible for the lattice-theoretic behavior of the operations $\Cap$ and $\Cup$ of an MV algebra.

It turns out that the relation $\preceq$ coincides with the usual partial order of $\mathscr{E}(\mathcal{H})$, induced by the class of all density operators of $\mathcal{H}$. Moreover,

$$E \Cap F = \begin{cases} E & \text{if} \quad E \preceq F \\ F & \text{otherwise} \end{cases}$$

and

$$E \Cup F = \begin{cases} E & \text{if} \quad F \preceq E \\ F & \text{otherwise} \end{cases}$$

The class of all $\lambda\mathbb{1}$ (where $\lambda \in [0, 1]$) determines a subalgebra of $\mathscr{E}(\mathcal{H})$, which is isomorphic to the standard MV algebra.

Let $\langle M, \oplus, {}^*, \mathbf{1}, \mathbf{0}\rangle$ be a QMV algebra. One can prove the following theorems (Giuntini, n.d.).

Theorem 1.1. The following properties hold:

(i) $a \odot b = b \odot a$.
(ii) $a \odot (b \odot c) = (a \odot b) \odot c$.
(iii) $a \odot a^* = \mathbf{0}$.
(iv) $a \odot \mathbf{0} = \mathbf{0}$.
(v) $a \odot \mathbf{1} = \mathbf{1}$.
(vi) $a \Cap \mathbf{1} = a = \mathbf{1} \Cap a$.
(vii) $a \Cap \mathbf{0} = \mathbf{0} = \mathbf{0} \Cap a$.
(viii) $a = a \Cap a$.
(ix) $(a \Cup b)^* = a^* \Cap b^*$.

(x) $(a \Cap b)^* = a^* \Cup b^*$.
(xi) If $a \preceq b$, then $a = b \Cap a$.

It should be noticed that, in general, $a = b \Cap a$ *does not* imply $a = a \Cap b$.

Theorem 1.2. The following properties hold:

(i) If $a \oplus b = \mathbf{0}$, then $a = b = \mathbf{0}$.
(ii) If $a \odot b = \mathbf{1}$, then $a = b = \mathbf{1}$.
(iii) If $a \Cup b = \mathbf{0}$, then $a = b = \mathbf{0}$.
(iv) If $a \Cap b = \mathbf{1}$, then $a = b = \mathbf{1}$.

Theorem 1.3 (Cancellation law). For any $a, b, c \in M$: if $a \oplus c = b \oplus c$, $a \preceq c^*$, and $b \preceq c^*$, then $a = b$.

Theorem 1.4. If $a \preceq b$, then $a^* \oplus b = \mathbf{1}$.

It should be noticed that, in general, $a^* \oplus b = \mathbf{1}$ does not imply $a \preceq b$.

Theorem 1.5. The following properties hold:

(i) If $a \preceq b$, then $b^* \preceq a^*$.
(ii) $a \preceq b$ iff $b = b \Cup a = a \Cup b$.
(iii) $a \Cap (b \Cup a) = a$.

Theorem 1.6. $\langle M, \preceq, {}^*, \mathbf{1}, \mathbf{0}\rangle$ is an involutive bounded poset.

Theorem 1.7. The following properties hold:

(i) If $a \preceq b$, then $\forall c \in M$: $a \Cap c \preceq b \Cap c$ (weak monotony of $\Cap$).
(ii) If $a \preceq b$, then $\forall c \in M$: $a \Cup c \preceq b \Cup c$ (weak monotony of $\Cup$).

It should be noticed that, in general, $a \Cap b \not\preceq a$, $a \not\preceq a \Cup b$, and $a \Cap b \not\preceq b \Cup a$.

Theorem 1.8 (Monotony of $\oplus$ and $\odot$). The following properties hold:

(i) If $a \preceq b$, then $\forall c \in M$: $a \oplus c \preceq b \oplus c$.
(ii) If $a \preceq b$, then $\forall c \in M$: $a \odot c \preceq b \odot c$.
(iii) If $a \preceq b$ and $c \preceq d$, then $a \oplus c \preceq b \oplus d$.
(iv) If $a \preceq b$ and $c \preceq d$, then $a \odot c \preceq b \odot d$.

Theorem 1.9. The following properties hold:

(i) $a \odot b \preceq a$.
(ii) $a \preceq a \oplus b$.
(iii) $a \odot b \preceq a \Cap b$, $a \odot b \preceq b \Cap a$.
(iv) $a \Cup b \preceq a \oplus b$, $b \Cup a \preceq a \oplus b$.

Theorem 1.10. The following conditions are equivalent:

(i) $\mathcal{M}$ is an MV algebra.
(ii) $\forall a, b \in M$: If $a^* \oplus b = \mathbf{1}$, then $a \leq b$.

Corollary 1.1. If $\mathcal{M}$ is a linear (or totally ordered) QMV algebra, then $\mathcal{M}$ is an MV algebra.

2. QUASILINEAR AND WEAKLY LINEAR QMV ALGEBRAS

In this section, we will introduce the notions of *quasilinearity* and *weak linearity.* It turns out that these two notions collapse into linearity whenever restricted to the class of all of MV algebras. We will prove that the equational class of all QMV algebras ($\mathcal{QMV}$) strictly includes the equational class generated by the class of all weakly linear QMV algebras [HSP(WLQMV)]. Finally, we will show that HSP(WLQMV) strictly includes the equational class generated by the class of all quasilinear QMV algebras [HSP(QLQMV)]. As to MV algebras, Chang (1957) proved that $\mathcal{MV}$ = HSP(LMV), where $\mathcal{MV}$ is the (equational) class of all MV algebras and HSP(LMV) is the variety generated by the class of all linear (= totally ordered) MV algebras.

Definition 2.1. A QMV algebra $\mathcal{M} = \langle M, \oplus, {}^*, \mathbf{1}, \mathbf{0}\rangle$ is said to be *quasilinear* (or *quasi-totally ordered*) iff $\forall a, b \in M$: if $a \not\leq b$, then $a \Cap b = b$.

In other words, a QMV algebra $\mathcal{M}$ is quasilinear iff the following holds:

$$a \Cap b = \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{otherwise} \end{cases}$$

One can easily check that an MV algebra $\mathcal{M}$ is quasilinear iff $\mathcal{M}$ is linear. Since we know that there are MV algebras that are not linear, we can conclude that not every QMV algebra is quasilinear. Both the standard MV algebra and the standard QMV algebra are quasilinear.

Lemma 2.1. Let $\mathcal{M}$ be a QMV algebra. The following conditions are equivalent:

(i) $\mathcal{M}$ is quasilinear.
(ii) $\forall a, b \in M$: if $a \oplus b \neq \mathbf{1}$, then $a < b^*$.
(iii) $\forall a, b, c \in M$: if $a \oplus c = b \oplus c \neq \mathbf{1}$, then $a = b$.

The notion of linearity can be furtherly weakened as follows.

Definition 2.2. A QMV algebra $\mathcal{M}$ is said to be *weakly linear* iff $\forall a, b \in M$: $a \Cap b = b$ or $b \Cap a = a$.

By the cancellation law, a QMV algebra is weakly linear iff $\forall a, b$: $a \oplus b^* = 1$ or $a^* \oplus b = 1$. Clearly, any MV algebra is weakly linear iff it is quasilinear iff it is linear. Every quasilinear QMV algebra is weakly linear, but not the other way around (as a counterexample see the QMV algebra $\mathcal{M}_{wl}$ of Fig. 2).

Theorem 2.1. $\mathbb{HSP}(\text{WLQMV}) \subset \mathcal{QMV}$.

Proof. In order to prove the theorem we will show that the equation

$$[(a \oplus b \odot b) \Cap (a \oplus b^* \odot b^*)]^* \oplus a = \mathbf{1} \qquad (*)$$

holds in every weakly linear QMV algebra, but fails in a particular QMV algebra.

Let $\mathcal{M}$ be a weakly linear QMV algebra. Then, $b \odot b = \mathbf{0}$ or $b^* \odot b^* = \mathbf{0}$. Suppose $b \odot b = \mathbf{0}$. Then,

$$lhs = [a \Cap (a \oplus b^* \odot b^*)]^* \oplus a = a^* \oplus a = \mathbf{1}$$

where *lhs* denotes the left-hand side of the equation (*). The proof is similar for $b^* \odot b^* = \mathbf{0}$.

We now prove that (*) does not hold in $\mathcal{QMV}$. Let us consider the orthomodular lattice $\mathcal{L}_{10}$ (Fig. 1). As proved in Giuntini (n.d.), every orthomodular lattice can be thought of as a QMV algebra, by taking $\oplus$ as the sup ($\sqcup$) and * as the orthocomplement $^\perp$.

Thus,

$$\begin{aligned} lhs &= [a \sqcup (b \sqcap b)] \Cap [a \sqcup (b^\perp \sqcap b^\perp)]^\perp \sqcup a \\ &= [(a \sqcup b) \Cap (a \sqcup b^*)]^* \sqcup a \\ &= (\mathbf{1} \Cap \mathbf{1})^* \sqcup a = a \neq \mathbf{1} \quad \blacksquare \end{aligned}$$

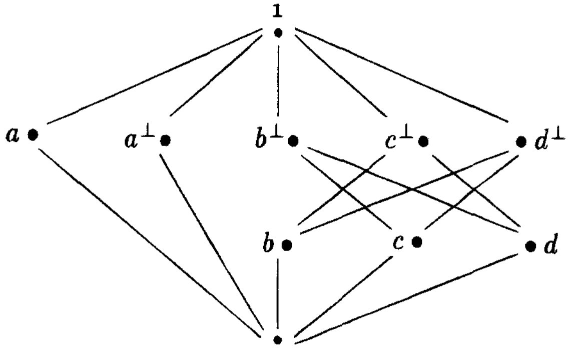

Fig. 1. $\mathcal{L}_{10}$.

Table I.

		$\oplus$
a	a	**1**
a	b	**1**
a	b^*	**1**
a^*	a^*	b
a^*	b	**1**
a^*	b^*	b
b	b	**1**
b	a	**1**
b	a^*	**1**
b^*	b^*	a
b^*	a	**1**
b^*	a^*	b

Theorem 2.2. HSP(QLQMV) $\subset$ HSP(WLQMV).

Proof. It suffices to show that the equation

$$(a^* \oplus a^*) \Cap (a \oplus a) = (a^* \Cap a) \oplus (a^* \Cap a) \qquad (**)$$

holds in every quasilinear QMV algebra, but fails in a particular weakly linear QMV algebra.

Let $\mathcal{M}$ be any quasilinear QMV algebra. Two cases are possible: (i) $a^* \preceq a$; (ii) $a^* \npreceq a$.

(i) By Theorem 1.4, $a \oplus a = \mathbf{1}$; hence $lhs = (a^* \oplus a^*) \Cap \mathbf{1} = a^* \oplus a^* = rhs$.

(ii) By Theorem 2.1, $a^* \oplus a^* = \mathbf{1}$, so that $(a^* \Cap a) = a$; thus, $lhs = a \oplus a = rhs$.

We now prove that (**) does not hold in the weakly linear QMV algebra $\mathcal{M}_{wl}$. In $\mathcal{M}_{wl}$, the operation $\oplus$, apart the obvious conditions, is defined as in Table I. $\mathcal{M}_{wl}$ can be represented as in Fig. 2. One can check that $\mathcal{M}_{wl}$ is a weakly linear QMV algebra. However, $\mathcal{M}_{wl}$ is not quasilinear, for $a^* \Cap a = (a^* \oplus a^*) \odot a = b \odot a = (a^* \oplus b^*)^* = b^*$.

The equation (**) does not hold in $\mathcal{M}_{wl}$ for $lhs = b \Cap \mathbf{1} \neq a = b^* \oplus b^* = rhs$. ■

Open Question: Are HSP(QLQMV) and HSP(WLQMV) finitely based?

3. LOCALLY FINITE QMV ALGEBRAS

In this section we will introduce the notions of *locally finite* and *Archimedean* QMV algebras and we will generalize some corresponding results of MV algebras.

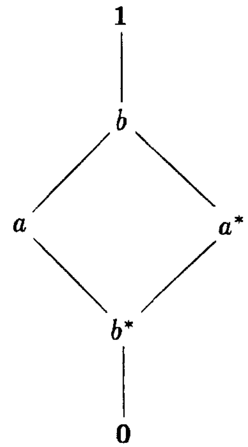

Fig. 2. $\mathcal{M}_{wl}$.

Definition 3.1. Let $\mathcal{M}$ be a QMV algebra. Let us define the following operations, for all $a \in M$ and for all $n \in \mathbb{N}$:

(i) $0 \cdot a = \mathbf{0}$, $(n + 1) \cdot a = n \cdot a \oplus a$.
(ii) $a^0 = \mathbf{1}$, $a^{n+1} = (a^n) \odot a$.

Definition 3.2. The *order* of an element a [briefly, $ord(a)$] is the least integer m, if it exists, s.t. $m \cdot a = \mathbf{1}$. If such an integer does not exist, $ord(a) = \infty$.

Clearly: $(n \cdot a)^* = (a^*)^n$; $(a^n)^* = (n \cdot a^*)$; $m \cdot (n \cdot a) = (m \cdot n) \odot a$; $a^{m+n} = (a)^m \odot (a)^n$; $a^{(m+n)} = (a^m)^n$.

Theorem 3.1 (Giuntini, n.d.). Let $\mathcal{M}$ be a QMV algebra. The following conditions are equivalent:

(i) $b = b \oplus a^*$.
(ii) $a \Cup b = \mathbf{1}$.
(iii) $a = a \oplus b^*$.
(iv) $b \Cup a = \mathbf{1}$.

Let us first recall a general theorem concerning QMV algebras.

Theorem 3.2. Let $\mathcal{M}$ be a QMV algebra. If $a \Cup b = \mathbf{1}$, then $\forall n \in \mathbb{N}$: $a^n \Cup b^n = \mathbf{1}$.

Proof. Let us suppose that $a \Cup b = \mathbf{1}$. By Theorem 3.1, $b = b \oplus a^*$. First of all, we prove that $a^n \Cup b = \mathbf{1}$.

By Theorem 3.1, it suffices to show that $b = b \oplus (a^n)^* = b \oplus (n \cdot a^*)$. We have

$$\begin{aligned} b \oplus (n \cdot a^*) &= b \oplus \underbrace{a^* \oplus \cdots \oplus a^*}_{n \text{ times}} \\ &= (b \oplus a^*) \oplus \underbrace{a^* \oplus \cdots \oplus a^*}_{n-1 \text{ times}} \\ &= b \oplus \underbrace{a^* \oplus \cdots \oplus a^*}_{n-1 \text{ times}} \qquad \text{(Th. 3.1)} \\ &\vdots \\ &= b \oplus a^* \\ &= b \end{aligned}$$

Thus, $a^n \oplus b = \mathbf{1}$. By Theorem 3.1, we have that $b \Cup a^n = \mathbf{1}$. Thus, by the previous argument, $b^n \Cup a^n = \mathbf{1}$, so that, by Theorem 3.1, $a^n \Cap b^n = \mathbf{1}$. ■

Theorem 3.3. Let $\mathcal{M}$ be a locally finite QMV algebra. If $ord(a \odot b) < \infty$, then $a \oplus b = \mathbf{1}$.

Proof. By hypothesis, $\exists n \in \mathbb{N}$ s.t. $n \cdot (a \odot b) = \mathbf{1}$. Hence $(a^* \oplus b^*)^n = \mathbf{0}$. By (QMV12), $(a^* \oplus b^*) \Cup (a + b) = \mathbf{1}$. By Theorem 3.2, $(a^* \oplus b^*)^n \Cup (a + b)^n = \mathbf{1}$. Since $(a^* \oplus b^*)^n = \mathbf{0}$, we obtain that $(a \oplus b)^n = \mathbf{1}$. By Theorem 1.2(ii), we can conclude that $a \oplus b = \mathbf{1}$. ■

By Theorem 3.3, it follows that $\forall a \in M$: if $ord(a) > 2$, then $ord(a \odot a) = \infty$.

Definition 3.3. A QMV algebra is said to be *locally finite* iff $\forall a \in M$ s.t. $a \neq \mathbf{0}$, $\exists n \in \mathbb{N}$ s.t. $n \cdot a = \mathbf{1}$.

Theorem 3.4. For any Hilbert space $\mathcal{H}$, the QMV algebra $\mathcal{E}(\mathcal{H})$ of all effects of $\mathcal{H}$ is locally finite.

Proof. Let us suppose, by contradiction, that $\exists E \in E(\mathcal{H})$ s.t. $E \neq \mathbb{O}$ and $\forall n \in \mathbb{N}$: $n \cdot E \neq \mathbb{1}$. An easy induction shows that

$$\forall n \in \mathbb{N}: \underbrace{E + \cdots + E}_{n \text{ times}} = n \cdot E < \mathbb{1}$$

Since $E \neq \mathbb{O}$, there exists a density operator D s.t. $\mathrm{Tr}(ED) = \lambda \neq 0$. Thus, $\exists k \in \mathbb{N}$ s.t. $k\lambda > 1$. By hypothesis and the induction argument, we have that

$$\underbrace{E + \cdots + E}_{k \text{ times}} = k \cdot E < \mathbb{1}$$

Thus,

$$\mathrm{Tr}(D(\underbrace{E + \cdots + E}_{k \text{ times}})) = k\,\mathrm{Tr}(DE) > 1$$

Contradiction. ■

Theorem 3.5. Every locally finite QMV algebra $\mathcal{M}$ is weakly linear.

Proof. We have to prove that $\forall a, b \in M$: $a^* \oplus b = \mathbf{1}$ or $a \oplus b^* = \mathbf{1}$. Let suppose that $a \oplus b^* \neq \mathbf{1}$. Then, $a^* \odot b \neq \mathbf{0}$. Since $\mathcal{M}$ is locally finite, $ord(a^* \odot b) < \infty$. By Theorem 3.3, we obtain that $a^* \oplus b = \mathbf{1}$. ■

As a corollary we have that every locally finite MV algebra is linear.

Since we know that there are linear MV algebras which are not locally finite, we can conclude that not every weakly linear QMV algebra is locally finite.

Definition 3.4. A QMV algebra is said to be *Archimedean* iff $\forall a, b$: if $\forall n \in \mathbb{N}$, $n \cdot a \preceq b$, then $a \odot b = \mathbf{a}$.

Theorem 3.6. Every quasilinear and Archimedean QMV algebra $\mathcal{M}$ is locally finite.

Proof. Let us suppose, by contradiction, that $\exists a \in M$ s.t. $a \neq \mathbf{0}$ and $\forall n \in \mathbb{N}$: $n \cdot a \neq \mathbf{1}$. We want to prove that $n \cdot a \preceq a^*$. Let suppose, on the contrary, that $n \cdot a \not\preceq a^*$. Since $\mathcal{M}$ is quasilinear, we have that $(n \cdot a) \Cap a^* = a^*$. This means, by the cancellation law, that $(n + 1) \cdot a = \mathbf{1}$, which contradicts the hypothesis that $ord(a) = \infty$. Thus, $\forall n \in \mathbb{N}$: $n \cdot a \preceq a^*$. Since $\mathcal{M}$ is Archimedean, we can conclude that $a = a \odot a^* = \mathbf{0}$, contradiction. ■

Open Question: Is every weakly linear and Archimedean QMV algebra locally finite?

Theorem 3.7. Every locally finite QMV algebra is Archimedean.

Proof. Let $a, b \in M$ and let us suppose that $\forall n \in \mathbb{N}$: $n \cdot a \preceq b$. If $a = \mathbf{0}$, then the theorem is proved. Thus, we can suppose that $a \neq \mathbf{0}$. Then, since $\mathcal{M}$ is locally finite, $\exists n \in \mathbb{N}$ s.t. $n \cdot a = \mathbf{1}$. This implies $b = \mathbf{1}$, so that $a = a \odot \mathbf{1} = a \odot b$. ■

Corollary 3.1. The standard QMV algebras are Archimedean.

Summing up:

MV Algebras

Linear $\Leftrightarrow$ Quasilinear $\Leftrightarrow$ Weakly Linear

$$\mathbb{HSP}(\text{WLMV}) = \mathbb{HPS}(\text{LMV}) = \mathcal{MV}$$

$$\begin{array}{ccc} \text{Archimedean} + \text{Linear} \Leftrightarrow & \text{Locally Finite} & \Rightarrow \text{Linear} \\ & \Downarrow & \\ & \text{Archimedean} & \end{array}$$

QMV Algebras

$$\text{Linear} \Rightarrow \text{Quasilinear} \Rightarrow \text{Weakly Linear}$$

$$\mathbb{HSP}(\text{QLQMV}) \subset \mathbb{PS}(\text{WLQMV}) \subset \mathcal{QMV}$$

$$\begin{array}{ccc} & \text{Weakly Linear} & \\ & \Uparrow & \\ \text{Archimedean} + \text{Quasilinear} \Rightarrow & \text{Locally Finite} & \not\Rightarrow \text{Quasilinear} \\ & \Downarrow & \\ & \text{Archimedean} & \end{array}$$

Open Question:

$$\text{Archimedean} + \text{Weakly Linear} \overset{?}{\Rightarrow} \text{Locally Finite}$$

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